Baby's First Diagrammatic Calculus for Quantum Information Processing

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Quantum computing

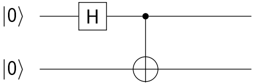
- Quantum computing is usually described using finite-dimensional Hilbert spaces and linear maps (or finite-dimensional C*-algebras and completely positive maps).
- Computing the matrix representation of quantum operations requires memory exponential in the number of input qubits.
- This is not a scalable approach for software applications related to quantum information processing (QIP).

Quantum computing

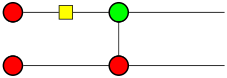
- Quantum computing is usually described using finite-dimensional Hilbert spaces and linear maps (or finite-dimensional C*-algebras and completely positive maps).
- Computing the matrix representation of quantum operations requires memory exponential in the number of input qubits.
- This is not a scalable approach for software applications related to quantum information processing (QIP).
- An alternative is provided by the *ZX-calculus* which is a sound, complete and universal diagrammatic calculus for equational reasoning about finite-dimensional quantum computing.

ZX-calculus

- The calculus is *diagrammatic* (some similarities to quantum circuits).
  - Example: Preparation of a Bell state.



$\mapsto$



- The ZX-calculus is *practical*. Used to study and discover new results in:
  - Quantum error-correcting codes [Chancellor, Kissinger, et. al 2016].
  - Measurement-based quantum computing [Duncan & Perdrix 2010].
  - (Quantum) foundations [Backens & Duman 2014].
  - and others...

ZX-calculus

- The ZX-calculus is *formal*.
  - Developed through the study of *categorical quantum mechanics*.
  - Rewrite system based on string diagrams of dagger compact closed categories.
  - Universal: Any linear map in **FdHilb** is the interpretation of some ZX-diagram D.
  - Sound: If $ZX \vdash D_1 = D_2$, then $[\![D_1]\!] = [\![D_2]\!]$.
  - Complete: If $[\![D_1]\!] = [\![D_2]\!]$, then $ZX \vdash D_1 = D_2$.

ZX-calculus

- The ZX-calculus is amenable to *automation* and *formal reasoning*.
  - Implemented in the Quantomatic proof assistant.

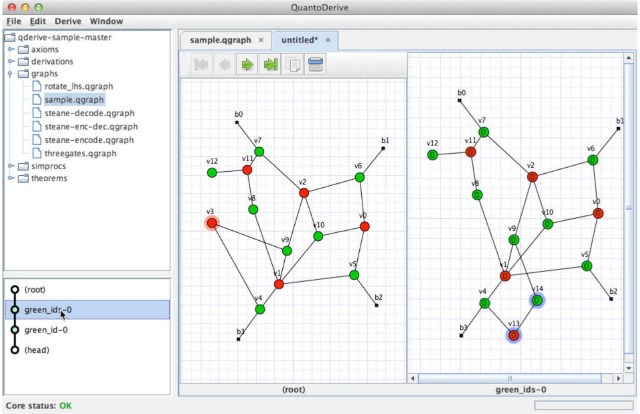


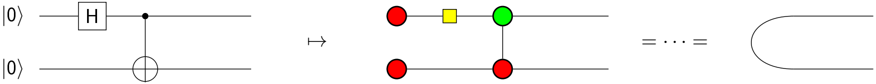
Figure: The quantomatic proof assistant.

ZX-calculus

- The ZX-calculus provides a different *conceptual perspective* of quantum information.

Example

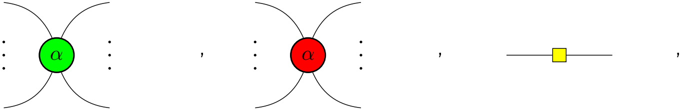
The Bell state is the standard example of an entangled state:



The diagrammatic notation clearly indicates this is not a separable state.

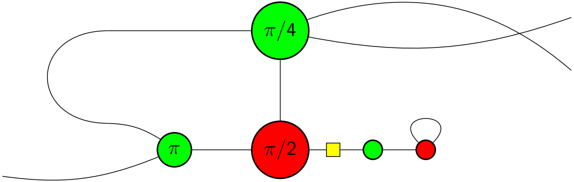
Syntax

A ZX-diagram is an open undirected graph constructed from the following generators:



where $\alpha \in [0, 2\pi)$.

Example



Syntax

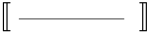
The ZX-calculus is an equational theory. Equality is written as $D_1 = D_2$:

Example

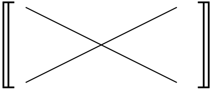
Normalisation

Remark: I ignore scalars and normalisation throughout the rest of the talk for brevity. But that can be handled by the language.

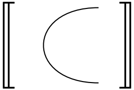
Semantics: wires



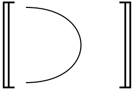
$$:= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$:= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

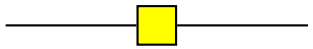


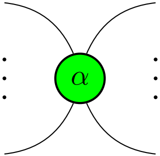
$$= |00\rangle + |11\rangle$$

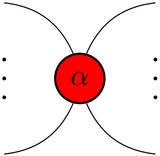


$$= \langle 00| + \langle 11|$$

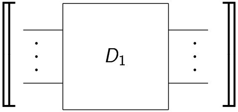
Semantics: spiders and hadamard

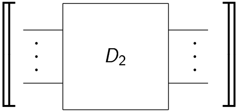
$[\![$  $]\!]$ $= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = H$

$[\![$  $]\!]$ $= \begin{cases} |0^m\rangle \mapsto |0^n\rangle \\ |1^m\rangle \mapsto e^{i\alpha}|1^n\rangle \\ \text{others} \mapsto 0 \end{cases}$

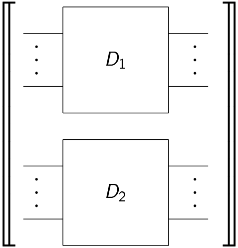
$[\![$  $]\!]$ $= \begin{cases} |+^m\rangle \mapsto |+^n\rangle \\ |-^m\rangle \mapsto e^{i\alpha}|-^n\rangle \\ \text{others} \mapsto 0 \end{cases}$

Semantics: tensors

If 

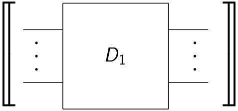
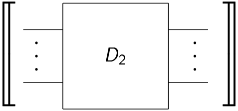
$= M_1$ and 

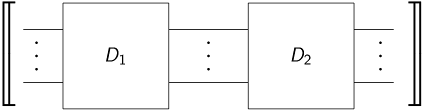
$= M_2$, then:



$= M_1 \otimes M_2$

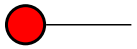
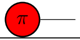
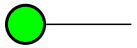
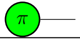
Semantics: composition

If  $= M_1$ and  $= M_2$, then:

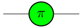
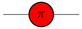
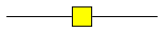
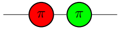
 $= M_2 \circ M_1$

By following these rules we can represent any linear map $f : \mathbb{C}^{2^m} \mapsto \mathbb{C}^n$ as a ZX-diagram (universality).

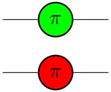
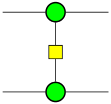
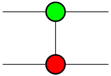
Example: Quantum States

State	ZX-diagram
$\lvert 0\rangle$	
$\lvert 1\rangle$	
$\lvert +\rangle$	
$\lvert -\rangle$	
$\lvert 00\rangle + \lvert 11\rangle$	

Example: unitary operations

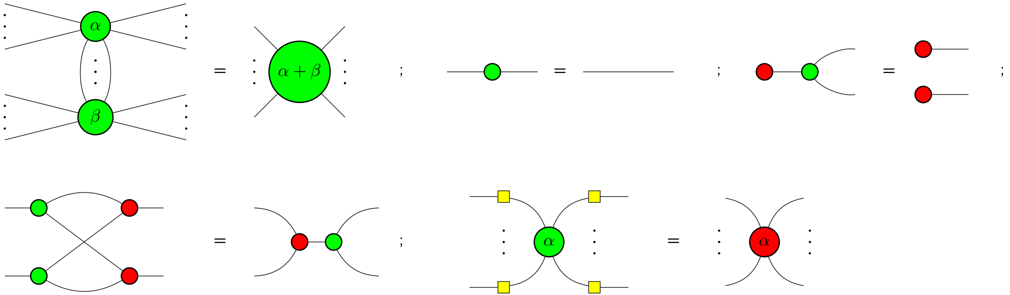
Unitary map	ZX-diagram
Z	
X	
H	
$Z \circ X$	

Example: 2-qubit gates

Unitary map	ZX-diagram
$Z \otimes X$	
$\wedge Z$	
CNOT	

Proof system

There are only 12 axioms. Five of them are:

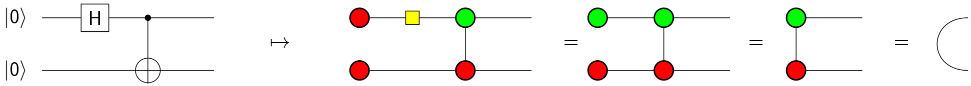


Remark: The color-swapped versions follow as *derived rules.*

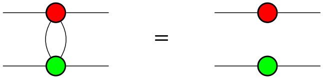
Remark: This rewrite system is sound and complete, i.e. no need for linear algebra:

$$ZX \vdash D_1 = D_2 \iff [\![D_1]\!] = [\![D_2]\!].$$

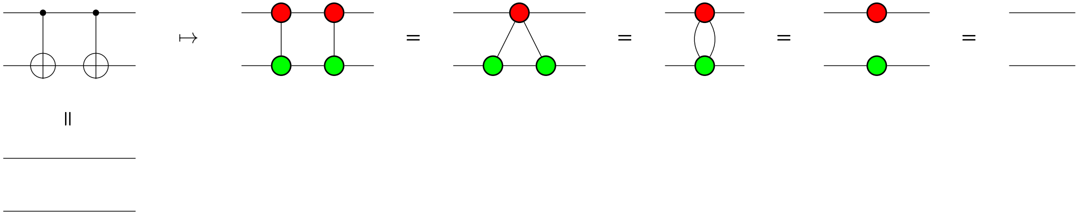
Example: Preparation of Bell state



Example: CNOT is self-adjoint

Lemma: 

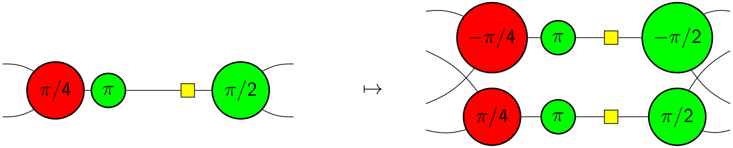
Then:



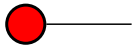
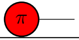
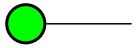
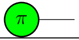
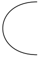
From Hilbert spaces to C*-algebras

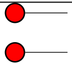
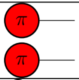
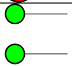
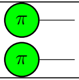
- So far we talked about pure state quantum mechanics (**FdHilb**).
- Next, we show how to model mixed-states (**FdCStar**).
- I omit some details for simplicity (see [Coecke & Kissinger, Picturing Quantum Processes]).
- The basic idea is to double up our diagrams and negate all angles in one of the copies (but there are other ways as well).

Example

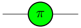
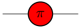
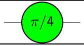
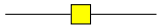
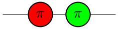


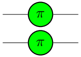
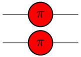
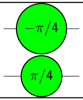
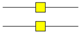
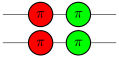
Example: Quantum States

State (**FdHilb**)	ZX-diagram
$\vert 0\rangle$	
$\vert 1\rangle$	
$\vert +\rangle$	
$\vert -\rangle$	
$\vert 00\rangle + \vert 11\rangle$	

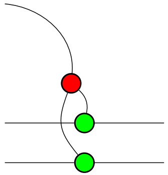
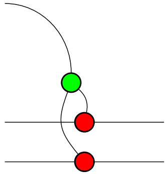
State (**FdCStar**)	ZX-diagram
$\vert 0\rangle\langle 0\vert$	
$\vert 1\rangle\langle 1\vert$	
$\vert +\rangle\langle +\vert$	
$\vert -\rangle\langle -\vert$	
$\vert 00\rangle\langle 00\vert + \vert 11\rangle\langle 11\vert$	

Example: unitary operations

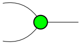
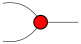
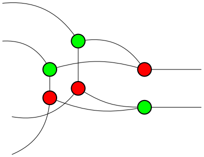
Unitary map (**FdHilb**)	ZX-diagram
Z	
X	
T	
H	
$Z \circ X$	

Unitary map (**FdCStar**)	ZX-diagram
Z	
X	
T	
H	
$Z \circ X$	

Example: conditional unitary operations

Conditional unitary	ZX-diagram
Z^b	
X^b	

Measuremens

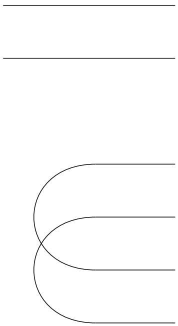
Measurement	ZX-diagram
Measurement in Z basis	
Measurement in X basis	
Measurement in Bell basis	

Example: quantum teleportation

0. A qubit owned by Alice.

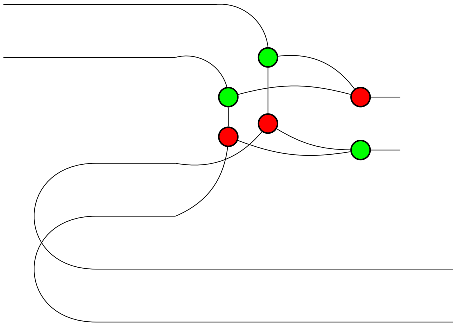
Example: quantum teleportation

1. Prepare Bell state.



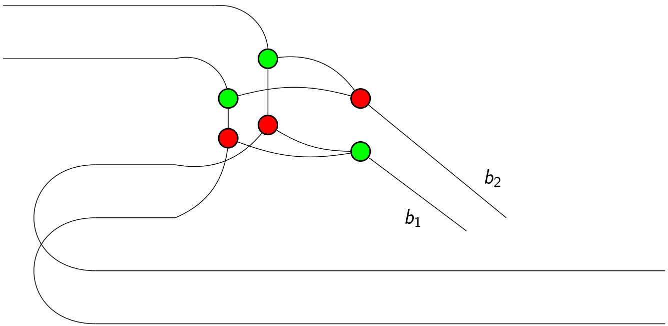
Example: quantum teleportation

2. Do Bell basis measurement on both of Alice's qubits.



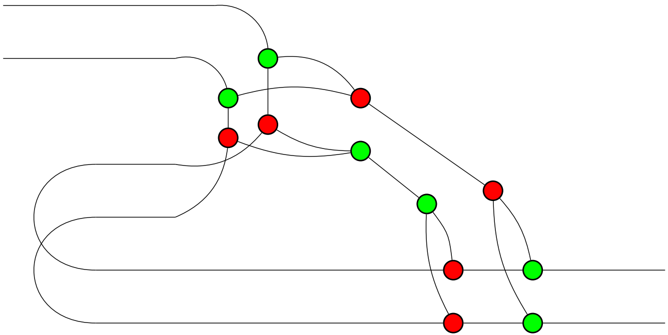
Example: quantum teleportation

3. Alice sends two bits (b_1, b_2) to Bob to inform him of measurement outcome.



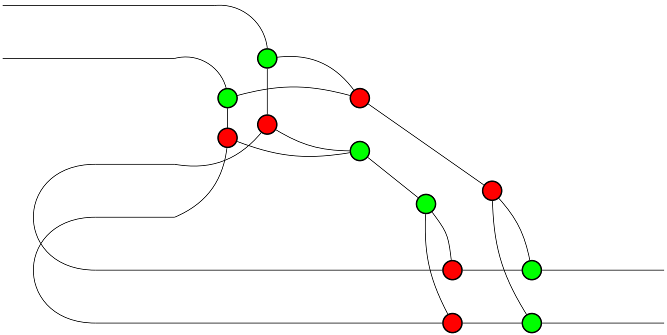
Example: quantum teleportation

4. Bob performs unitary correction $X^{b_1} \circ Z^{b_2}$.



Example: quantum teleportation

We can now prove the correctness of the teleportation protocol in the ZX-calculus:



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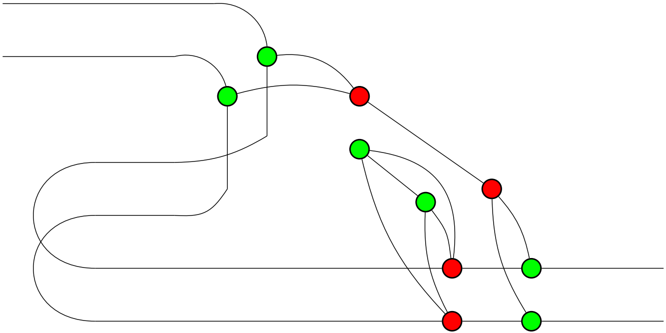
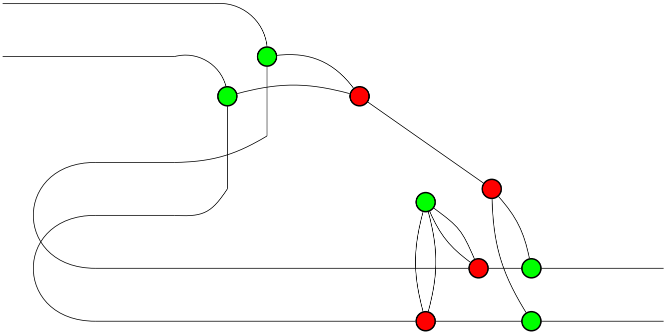
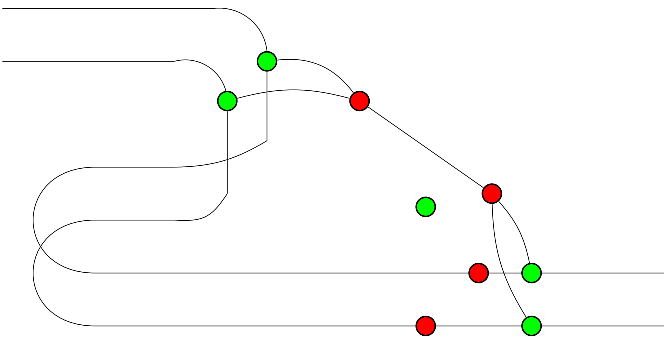
Example: quantum teleportation

$=$  $=$

Example: quantum teleportation

$=$  $=$

Example: quantum teleportation

$=$  $=$

Example: quantum teleportation

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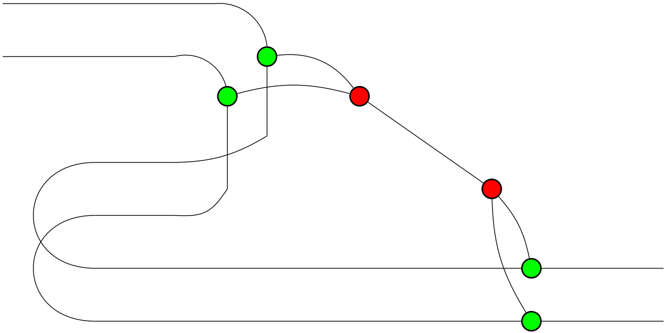

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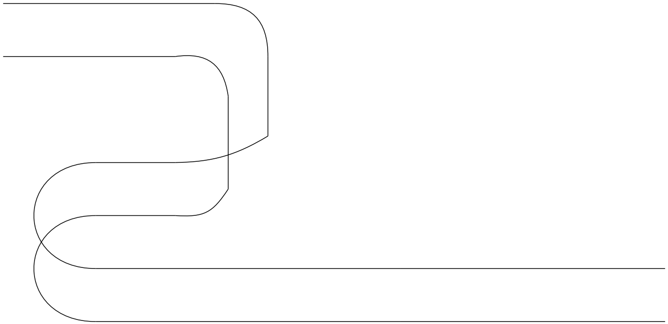
Example: quantum teleportation

$=$  $=$

Example: quantum teleportation

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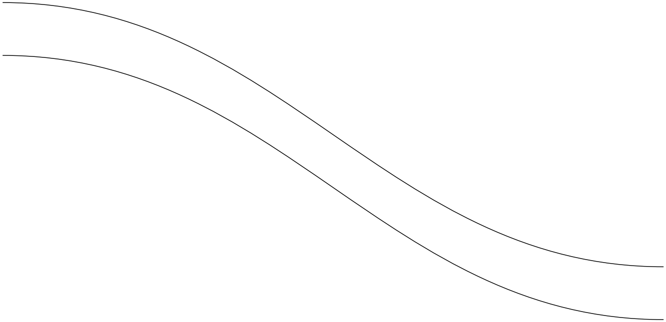
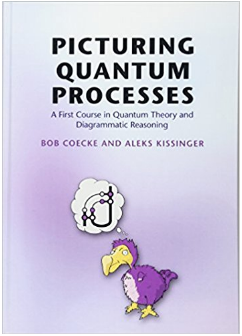


Therefore, teleportation works as expected.

Conclusion

- The ZX-calculus is a sound and complete alternative to linear algebra for *finite-dimensional* quantum information processing.
- Great potential for formal methods, verification and computer-aided reasoning.
- Useful tool for studying QIP.
- If you want to learn more, check out the book (contains outdated and incomplete version of ZX):



Thank you for your attention!